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Quark-anti-quark potential in $\mathcal{N} = 4$ SYM

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ABSTRACT: We construct a closed system of equations describing the quark-anti-quark potential at any coupling in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. It is based on the Quantum Spectral Curve method supplemented with a novel type of asymptotics. We present a high precision numerical solution reproducing the classical and one-loop string predictions very accurately. We also analytically compute the first 7 nontrivial orders of the weak coupling expansion.

Moreover, we study analytically the generalized quark-anti-quark potential in the limit of large imaginary twist to all orders in perturbation theory. We demonstrate how the QSC reduces in this case to a one-dimensional Schrodinger equation. In the process we establish a link between the Q-functions and the solution of the Bethe-Salpeter equation.

KEYWORDS: AdS-CFT Correspondence, Integrable Field Theories

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1 Introduction

During the recent years $\mathcal{N} = 4$ SYM in 4d has been studied very intensively. One of the reasons it attracted so much attention is the existence of a well established AdS/CFT dual. At the same time it is also a beautiful theory by itself which has numerous links with the more realistic QCD.

One of the first predictions of AdS/CFT was the strong coupling limit of the potential between two heavy charged particles or “quarks” which is represented by a pair of anti-parallel Wilson lines separated by distance r [1, 2]. The potential is inversely proportional to the separation r due to conformal symmetry of the theory, with the strength of the

interaction depending on the gauge coupling g_{YM} . In the planar limit $N_c \rightarrow \infty$ the potential is a highly non-trivial function of the 't Hooft coupling $\lambda = g_{YM}^2 N_c$,

$$V(\lambda, r) = -\frac{\Omega(\lambda)}{r}. \quad (1.1)$$

Currently the function $\Omega(\lambda)$ is known at 3 loops at weak coupling [1, 3–8] and at one loop at strong coupling [9–12]. In fact even at low orders the weak coupling expansion is rather involved and requires using a nontrivial low-energy effective theory. One can further generalize this observable by introducing an extra parameter θ , which may be associated with relative flavors of the particles. The particle flavor enters through the unit vector \vec{n} in the expression for the Maldacena-Wilson line

$$\text{Pexp} \left[\int (iA_\mu \dot{x}^\mu + \vec{\Phi} \cdot \vec{n} |\dot{x}|) \right]. \quad (1.2)$$

The parameter θ is the angle between these vectors \vec{n} for the two antiparallel lines. The expectation value of the pair of the Maldacena-Wilson lines is related to the potential as

$$\langle W \rangle \simeq e^{\frac{T\Omega(\lambda, \theta)}{r}} \quad (1.3)$$

where $T \gg r$ is the extent of the lines.

In this paper we study this important observable $\Omega(\lambda, \theta)$ intensively using the integrability-based Quantum Spectral Curve method introduced for local operators in [13, 14] and generalized for a subclass of Wilson lines in [15]. We show how the results of [15] can be used to get a closed system of equations describing $\Omega(\lambda, \theta)$ exactly in the whole range of the parameters λ and θ . We find the analytic weak coupling expansion up to 7 loops and also build a numerically-exact function interpolating from weak to strong coupling regime. Finally, we study analytically the limit $\theta \rightarrow i\infty$ (with $\lambda e^{-i\theta}$ fixed) to all orders in the 't Hooft coupling.¹ We demonstrate how the Schrödinger equation arising from resummation of the ladder diagrams in this limit appears from the Quantum Spectral Curve.

2 Quantum Spectral Curve for the quark-anti-quark potential

The configuration of two anti-parallel Wilson lines is closely related to a configuration where two straight lines meet at a cusp where they form an angle ϕ [17]. Indeed, the two setups are linked by the plane to cylinder transformation where the cusp point is mapped to infinity. In this picture the distance between the lines is given by $r = \phi - \pi$. When ϕ tends to π the curvature of the cylinder becomes irrelevant and one recovers the flat space quark-anti-quark potential.

In [8, 18, 19] it was shown that the anomalous dimension of the cusped Maldacena-Wilson line admits an integrability-based description in terms of an infinite system of integral equations (known as Thermodynamic Bethe Ansatz equations). This anomalous

¹A similar limit in the γ -deformed SYM was recently considered in [16].

dimension depends on 3 parameters: θ, ϕ and the coupling $g = \frac{\sqrt{\lambda}}{4\pi}$. Subsequently a much simpler description in terms of the Quantum Spectral Curve (QSC) was found [15] which we use here.

In this section we review the QSC construction for this observable and introduce important notation relevant for the rest of the paper.

2.1 QSC for a Wilson line with a cusp

The Quantum Spectral Curve [13, 14] is a set of finite difference equations on Q-functions supplemented with very precise analytical properties. There are $4 + 4$ basic Q-functions denoted as \mathbf{P}_a , $a = 1, \dots, 4$ and \mathbf{Q}_i , $i = 1, \dots, 4$. They can be roughly associated with S^5 and AdS_5 degrees of freedom correspondingly. For different observables with different quantum numbers one should adjust accordingly the asymptotics and sometimes modify the analytical properties. In the rest of the section we review the QSC construction proposed in [15] for the anomalous dimension Δ of the cusped Wilson line which is a function of two angles ϕ, θ and the coupling g .

The S^5 part of the QSC can be efficiently parameterized in terms of two functions \mathbf{f} and \mathbf{g} of the spectral parameter u analytic everywhere except the cut $[-2g, 2g]$ as follows [15]:

$$\begin{aligned} \mathbf{P}_1(u) &= +\epsilon u^{1/2} e^{+\theta u} \mathbf{f}(+u), \\ \mathbf{P}_2(u) &= -\epsilon u^{1/2} e^{-\theta u} \mathbf{f}(-u), \\ \mathbf{P}_3(u) &= +\epsilon u^{1/2} e^{+\theta u} \mathbf{g}(+u), \\ \mathbf{P}_4(u) &= +\epsilon u^{1/2} e^{-\theta u} \mathbf{g}(-u). \end{aligned} \quad (2.1)$$

We have to impose $\mathbf{f} \simeq 1/u$ and $\mathbf{g} \simeq u$ for large u . For this normalization the prefactor ϵ is fixed to be

$$\epsilon = \sqrt{\frac{i}{2}} \frac{\cos \theta - \cos \phi}{\sin \theta}. \quad (2.2)$$

It is convenient to resolve the cut $[-2g, 2g]$ by introducing the variable x

$$x(u) = \frac{u + \sqrt{u - 2g}\sqrt{u + 2g}}{2g}, \quad u = (x + 1/x)g. \quad (2.3)$$

This function maps the complex plane into the exterior of the unit circle $|x| > 1$. As a regular function of x at least for $|x| > 1$, the functions \mathbf{f} and \mathbf{g} can be written in terms of the Laurent expansion coefficients

$$\mathbf{f}(u) = \frac{1}{gx} + \sum_{n=1}^{\infty} \frac{g^{n-1} A_n}{x^{n+1}}, \quad \mathbf{g}(u) = \frac{u^2 + B_0 u}{gx} + \sum_{n=1}^{\infty} \frac{g^{n-1} B_n}{x^{n+1}}. \quad (2.4)$$

The first few coefficients encode the information about the AdS charges and twists, i.e. Δ and ϕ , via the relations [15]

$$A_1 g^2 - B_0 = -\frac{2 \cos \theta \cos \phi + \cos(2\theta) - 3}{2 \sin \theta (\cos \theta - \cos \phi)}, \quad (2.5)$$

$$\begin{aligned} \Delta^2 &= \frac{(\cos \theta - \cos \phi)^3}{\sin \theta \sin^2 \phi} \left[A_3 g^6 + \frac{A_1^2 g^4 (1 - \cos \theta \cos \phi)}{\sin \theta (\cos \theta - \cos \phi)} - A_2 g^4 \cot \theta - g^2 (B_0 + B_1 + \cot \theta) \right. \\ &\quad \left. - A_1 g^2 \left(A_2 g^4 - 2g^2 + \frac{1}{\sin^2 \theta} \right) \right]. \end{aligned} \quad (2.6)$$

We also note that the coefficients A_n and B_n are real and scale at weak coupling as $\mathcal{O}(g^0)$. Their leading weak coupling behavior can be deduced from [15] and is given in appendix A.

The AdS_5 constituents of the QSC, i.e. the \mathbf{Q}_i , are completely fixed in terms of \mathbf{P}_a . The procedure is the following: first one finds the “transition functions” $Q_{a|i}$ from the finite difference equation

$$Q_{a|i}(u + \frac{i}{2}) - Q_{a|i}(u - \frac{i}{2}) = -\mathbf{P}_a(u)\mathbf{P}^b(u)Q_{b|i}(u + \frac{i}{2}) \quad (2.7)$$

where $\mathbf{P}^a = \chi^{ab}\mathbf{P}_b$ and the only non-zero elements of the constant matrix χ^{ab} are $\chi^{41} = -\chi^{32} = \chi^{23} = -\chi^{14} = 1$. The finite difference equation (2.7) has 4 independent solutions labeled by the index $i = 1, \dots, 4$. It is always possible to impose analyticity of $Q_{a|i}(u - \frac{i}{2})$ above the real axis which we assume to be done. Once $Q_{a|i}$ are found, \mathbf{Q}_i are simply given by

$$\mathbf{Q}_i(u) = -\mathbf{P}^a(u)Q_{a|i}(u + \frac{i}{2}) . \quad (2.8)$$

Equivalently these relations can be reformulated [20] in a form of a 4th order finite difference equation for \mathbf{Q}_i with the coefficients built from \mathbf{P}_a

$$\begin{aligned} \mathbf{Q}_i^{[+4]}D_0 - \mathbf{Q}_i^{[+2]}\left[D_1 - \mathbf{P}_a^{[+2]}\mathbf{P}^{a[+4]}D_0\right] + \mathbf{Q}_i\left[D_2 - \mathbf{P}_a\mathbf{P}^{a[+2]}D_1 + \mathbf{P}_a\mathbf{P}^{a[+4]}D_0\right] \\ - \mathbf{Q}_i^{[-2]}\left[\bar{D}_1 + \mathbf{P}_a^{[-2]}\mathbf{P}^{a[-4]}\bar{D}_0\right] + \mathbf{Q}_i^{[-4]}\bar{D}_0 = 0 \end{aligned} \quad (2.9)$$

where D_n, \bar{D}_n are some nice combinations of \mathbf{P} ’s given in appendix B. As a 4th order equation it has 4 independent solutions which are precisely the \mathbf{Q}_i . One can show that the relations (2.6) and (2.5) imply the following large u asymptotics for \mathbf{Q}_i

$$\mathbf{Q}_1 \sim u^{1/2+\Delta}e^{u\phi}, \quad \mathbf{Q}_2 \sim u^{1/2+\Delta}e^{-u\phi}, \quad \mathbf{Q}_3 \sim u^{1/2-\Delta}e^{u\phi}, \quad \mathbf{Q}_4 \sim u^{1/2-\Delta}e^{-u\phi} . \quad (2.10)$$

This is in fact how (2.6) and (2.5) were derived. Those 4 distinguished asymptotics allow to choose the basis of solutions $\{\mathbf{Q}_i\}$ uniquely up to a normalization. The functions \mathbf{Q}_i are analytic in the upper half plane and have a cut $[-2g, 2g]$ on the real axis as well as more cuts below (as can be deduced from the equation (2.9)). A non-trivial new condition, which in fact allows to close the equations and fix the coefficients A_n and B_n uniquely, concerns the behavior of \mathbf{Q}_i on the cut $[-2g, 2g]$. To describe it we introduce

$$\mathbf{q}_i = \mathbf{Q}_i u^{-1/2} \quad (2.11)$$

and denote by $\tilde{\mathbf{q}}_i$ the analytic continuation of \mathbf{q}_i under the cut on the real axis. Then according to [15]

$$\tilde{\mathbf{q}}_1(u) = \mathbf{q}_1(-u) \quad (2.12)$$

$$\tilde{\mathbf{q}}_2(u) = \mathbf{q}_2(-u) \quad (2.13)$$

$$\tilde{\mathbf{q}}_3(u) = a_1 \sinh(2\pi u)\mathbf{q}_2(-u) + \mathbf{q}_3(-u) \quad (2.14)$$

$$\tilde{\mathbf{q}}_4(u) = a_2 \sinh(2\pi u)\mathbf{q}_1(-u) + \mathbf{q}_4(-u) . \quad (2.15)$$

It was noticed in [15] that it is sufficient to impose the first two equations in (2.12) only.

In the next section we discuss what happens in the singular limit $\phi \rightarrow \pi$ and derive a closed system of equations describing directly the potential $\Omega(\lambda, \theta)$.

2.2 QSC for the quark-anti-quark potential

In this paper we focus on the particularly important limit $\phi \rightarrow \pi$ when the Wilson line with a cusp is related to a pair of anti-parallel lines. In this limit we expect the anomalous dimension Δ to diverge as

$$\Delta = -\frac{\Omega(\lambda)}{\pi - \phi} + \mathcal{O}((\pi - \phi)^0) \quad (2.16)$$

where $\Omega(\lambda)$ is a positive quantity (for real θ). As the anomalous dimension diverges we should expect a drastic change in the large u asymptotics of \mathbf{Q}_i , which for finite Δ is given by (2.10). To get some intuition about what happens we take $\phi = \pi - \epsilon$ with ϵ being small, so the asymptotics becomes

$$\mathbf{q}_1 \sim e^{\pi u} \exp \left[-u\epsilon - \frac{\Omega}{\epsilon} \log u \right]. \quad (2.17)$$

We see that the last term in the second factor explodes for u fixed and the asymptotics does not make sense. What happens is that the subleading coefficients become bigger in this limit in order to make the result finite. However, if we scale u to infinity while sending $\epsilon \rightarrow 0$ we should be able to suppress the subleading in $1/u$ terms. The guiding principle is to try to balance the two terms in the square brackets, which is the case for $u \sim \Omega/\epsilon^2$ (treating $\log u$ as a constant compared to \sqrt{u}) or in other words for $\epsilon = c\sqrt{\Omega}/\sqrt{u}$ this results in

$$\log \mathbf{q}_1 \sim +\pi u - c\sqrt{\Omega u} + \mathcal{O}(u^0). \quad (2.18)$$

The positive constant c cannot be determined from this heuristic argument and it will be shown below to be equal to $\sqrt{8}$. Similar considerations for $\mathbf{q}_2, \mathbf{q}_3$ and \mathbf{q}_4 lead to

$$\log \mathbf{q}_2 \sim -\pi u + ic\sqrt{\Omega u} + \mathcal{O}(u^0), \quad (2.19)$$

$$\log \mathbf{q}_3 \sim +\pi u - ic\sqrt{\Omega u} + \mathcal{O}(u^0), \quad (2.20)$$

$$\log \mathbf{q}_4 \sim -\pi u + c\sqrt{\Omega u} + \mathcal{O}(u^0). \quad (2.21)$$

To get the precise value of the coefficient c and derive the asymptotics rigorously, we have to analyze the limit of \mathbf{P}_a when $\phi \rightarrow \pi$. One could expect that \mathbf{P}_a behave smoothly in this limit as they describe the S^5 part which is relatively isolated from the twist ϕ in AdS_5 . It can be also seen from (2.5) and (2.6) that we can consistently assume the coefficients A_n and B_n in \mathbf{P} 's to remain finite when $\phi \rightarrow \pi$, giving

$$\epsilon = \sqrt{\frac{i \cos \theta + 1}{2 \sin \theta}}, \quad B_0 = A_1 g^2 - \frac{2 - \cos \theta}{\sin \theta}, \quad (2.22)$$

$$\begin{aligned} \Omega^2 = & \frac{g^2 \cot^3 \frac{\theta}{2}}{2} [2 \sin \theta (A_3 g^4 \sin \theta - A_2 g^2 \cos \theta - B_1 \sin \theta - 2 \cos \theta + 2) \\ & + 2A_1^2 g^2 \sin \theta + A_1 (-2A_2 g^4 \sin^2 \theta - g^2 \cos(2\theta) + g^2 - 2)] . \end{aligned} \quad (2.23)$$

This allows to find the asymptotics of \mathbf{q}_i using the 4th order Baxter equation (2.9) in which we expand the coefficients at large u . The expressions we get are lengthy, and

for illustration purposes let us drop some of the terms which do not affect the leading asymptotics, leaving the following equation:

$$\begin{aligned} \mathbf{q}(u) \left(\frac{-\frac{2\Omega^2}{3} - 1}{u^2} + 1 \right) + \left(-\frac{2}{3u^2} + \frac{i}{3u} + \frac{2}{3} \right) \mathbf{q}(u+i) + \left(-\frac{1}{6u^2} + \frac{i}{6u} + \frac{1}{6} \right) \mathbf{q}(u+2i) \\ + \left(-\frac{2}{3u^2} - \frac{i}{3u} + \frac{2}{3} \right) \mathbf{q}(u-i) + \left(-\frac{1}{6u^2} - \frac{i}{6u} + \frac{1}{6} \right) \mathbf{q}(u-2i) = 0. \end{aligned}$$

While the coefficients in this equation are simple, the asymptotics of its four solutions is quite nontrivial. It turns out to have indeed the form anticipated above in (2.18) as we get

$$\mathbf{q}_i = M_i u^{1/4} e^{\pm \pi u + \alpha_i \sqrt{u}} (1 + \mathcal{O}(1/u)) \quad (2.24)$$

where

$$\alpha_1 = -\sqrt{8\Omega}, \quad \alpha_2 = +i\sqrt{8\Omega}, \quad \alpha_3 = -i\sqrt{8\Omega}, \quad \alpha_4 = +\sqrt{8\Omega}. \quad (2.25)$$

Expanding the Baxter equation to higher orders in u and keeping all the terms, we found the following expansion for the solution:

$$\mathbf{q}_i = M_i u^{1/4} e^{\pm \pi u + \alpha_i \sqrt{u}} \left(1 + \sum_{n=1}^{\infty} \frac{d_n}{(\alpha_i)^n u^{n/2}} \right) \quad (2.26)$$

This rather surprising asymptotics is a key result which supplements the QSC functional equations.

A natural way to fix the normalization of \mathbf{Q}_i (which we will use in this paper, e.g. in appendix D) is to impose that the matrix $Q_{a|i}$ preserves the constant matrix χ^{ab} , i.e.

$$Q_{a|i} \chi^{ab} Q_{b|j} \chi^{jk} = \delta_i^k \quad (2.27)$$

This leads to

$$iM_1 M_4 = M_2 M_3 = \sqrt{2} \frac{\cos^4(\theta/2)}{\Omega^{3/2}}. \quad (2.28)$$

We conclude that the quark-anti-quark potential is described by QSC with a novel type of asymptotics of the Q-functions containing non-integer powers of the spectral parameter u in the exponent. These asymptotics together with the general relations from the previous section form a closed system of equations applicable at all values of the coupling g and the twist θ .

Despite the anomalous dimension Δ of the cusped Wilson line being infinite at $\phi = \pi$, we managed to reformulate the QSC equations in such a way that they only include the finite residue $\Omega(\lambda, \theta)$ and got rid of the auxiliary parameter ϕ completely. In the following sections we will solve these equations both analytically at weak coupling to a high order and numerically in a wide range of the coupling. We will be also able to demonstrate how in a special limit the QSC reduces to the Schrödinger equation of [1, 4] resumming the ladder diagrams to all orders in perturbation theory.

3 Weak coupling

In this section we show how to solve the equations from the previous section perturbatively at weak coupling. We will see that the weak coupling limit is rather nontrivial and contains qualitatively new features compared to all other perturbative expansions of the Quantum Spectral Curve studied previously [14, 20–22].

3.1 Different scales and structure of the expansion

The weak coupling limit is more involved in the present case as it depends on the scaling of the spectral parameter u . The situation here is similar to the conventional perturbation theory where in order to compute the quark-anti-quark potential one has to work with an effective theory resumming soft contributions. We also note that the limits $\phi \rightarrow \pi$ and $g \rightarrow 0$ do not commute with each other and it is crucial to have a closed system of equations directly at $\phi = \pi$ in order to get a sensible weak coupling expansion.

Another key feature of the weak coupling calculation is that the limits $g \rightarrow 0$ and $u \rightarrow \infty$ do not commute. The reason for this is that Ω , appearing in the asymptotics (2.26), goes to zero as g^2 . In this case one should expect the following three natural scales

$$\begin{aligned} \text{scale 1 : } & u \rightarrow \infty \quad \text{before } g \rightarrow 0 \\ \text{scale 2 : } & g \rightarrow 0 \quad \text{with } v \equiv 8u\Omega \text{ fixed} \\ \text{scale 3 : } & g \rightarrow 0 \quad \text{then } u \rightarrow \infty \end{aligned}$$

In the scale 1 we are in the regime where the asymptotics (2.26) is still valid. The scale 2 is natural to consider as in the asymptotics (2.26) u appears in this combination with Ω . In the scale 3 we are in the usual perturbative regime of the QSC studied intensively in [14, 20–22] and we should expect the usual expansion of the Q-functions in terms of η -functions introduced in [23]. These η -functions are defined as²

$$\eta_{s_1, \dots, s_k}(u) = \sum_{n_1 > n_2 > \dots > n_k \geq 0} \frac{1}{(u + in_1)^{s_1} \dots (u + in_k)^{s_k}}. \quad (3.1)$$

At large u however these functions can only give terms of the type $u^n \log^m u$, which are very different from the scale 1. The intermediate scale 2 should match the two regimes corresponding to scales 1 and 3. This regime plays an important role as it allows to identify correctly \mathbf{q}_1 and \mathbf{q}_2 in the scale 3 and distinguish them from \mathbf{q}_3 and \mathbf{q}_4 , for which the analyticity condition on the cut $[-2g, 2g]$ given by (2.12) is different.

Thus, before we can use (2.12) and fix the coefficients A_n and B_n in the expression for \mathbf{P}_a , we have to pass through the regime with finite $v \equiv 8\Omega u$. Fortunately, in this regime the finite difference equation (2.9) on \mathbf{q}_i (related to \mathbf{Q}_i via (2.11)) simplifies into a 4th order differential equation which we can solve systematically order by order in g . Its solution provides a bridge between scale 1 and scale 3 by interpolating between the exponential and

²In some cases the sum could be divergent, we regularize it as in [21, 23] so that e.g. $\eta_1(u) = i\psi(-iu)$.

power-like with logs asymptotics. We will first demonstrate this procedure at the leading order in the coupling and then present our result to a high order in perturbation theory.³

To study the 2nd scale we start from the 4th order Baxter equation (2.9) and expand it at large u (notice that in the 2nd scale u is large as it is $\sim 1/g^2$). By doing this we obtain a finite difference equation of the form

$$\mathbf{q}_i(u) \left(1 + \frac{C_0}{u^2} + \dots \right) = \frac{\mathbf{q}_i(u+2i)}{6} \left(1 + \frac{i}{u} + \frac{C_2}{u^2} + \dots \right) + \frac{2\mathbf{q}_i(u+i)}{3} \left(1 + \frac{i}{2u} + \frac{C_1}{u^2} + \dots \right) \\ + \frac{\mathbf{q}_i(u-2i)}{6} \left(1 - \frac{i}{u} + \frac{\bar{C}_2}{u^2} + \dots \right) + \frac{2\mathbf{q}_i(u-i)}{3} \left(1 - \frac{i}{2u} + \frac{\bar{C}_1}{u^2} + \dots \right)$$

where C_n and the sub-leading coefficients are some explicit combinations of A_n and B_n . Next we use that $u = v/(8\Omega)$ where $\Omega \sim g^2$ and introduce a smooth function $f(v)$ such that $\mathbf{q}(u) = e^{\pm\pi u} f(8\Omega u)$ to obtain

$$f^{(4)} + \frac{2f^{(3)}}{v} - \frac{f}{16v^2} + 8\hat{g}^2 \frac{f''}{v^2} + \mathcal{O}(g^4) = 0 \quad (3.2)$$

where $\hat{g} = g \cos(\frac{\theta}{2})$. Fortunately, we can solve this equation analytically! At the leading order in g its 4 independent solutions are given by four different types of Bessel functions,

$$\sqrt{v} K_1(\sqrt{v}), \quad \sqrt{v} Y_1(\sqrt{v}), \quad \sqrt{v} I_1(\sqrt{v}), \quad \sqrt{v} J_1(\sqrt{v}). \quad (3.3)$$

Next we notice that the first solution should be related to \mathbf{q}_1 simply because its large v asymptotics matches precisely the asymptotics (2.26) of \mathbf{q}_1 :

$$f_1(v) \equiv \sqrt{v} K_1(\sqrt{v}) \simeq \sqrt{\frac{\pi}{2}}^{1/4} \sqrt{8\Omega u} e^{-\sqrt{8\Omega u}}. \quad (3.4)$$

We note that since this is one of the decaying “small” solutions this identification is non-ambiguous.

At the higher orders in g the equation (3.2) gets corrected. In general one would have to solve (3.2) using perturbation theory, involving Green’s function and multiple integrations. However, we found a much simpler procedure, which works magically up to at least g^{10} order. One can simply build an ansatz for the corrected solution as a linear combination of $v^{(1/2-m)} \partial_\nu^n K_\nu(\sqrt{v})|_{\nu=1}$ for integer m and n . So for instance at g^2 order we simply get

$$f_1(v) = \sqrt{v} K_1(\sqrt{v}) - 8\hat{g}^2 \sqrt{v} K_1^{(1,0)}(\sqrt{v}) + \mathcal{O}(g^4). \quad (3.5)$$

Having an explicit form of the solution in the scale 2, we can get information about the behavior of \mathbf{q}_1 in the scale 3. For that we expand (3.5) at small v ,

$$f_1(v) = 1 + \frac{1}{4}v \left(\log \frac{v}{4} + 2\gamma - 1 \right) + \frac{1}{64}v^2 \left(2 \log \frac{v}{16} + 4\gamma - 5 \right) + \mathcal{O}(v^3) \\ + 4\hat{g}^2 \left(\log \frac{v}{4} + 2\gamma \right) + \hat{g}^2 v \left(\log \frac{v}{4} + 2\gamma - 2 \right) + \mathcal{O}(g^2 v^2) + \mathcal{O}(g^4). \quad (3.6)$$

³At high orders to simplify intermediate expressions we used the HPL Mathematica package [24, 25] and the package for working with multiple zeta values provided with the paper [23].

We see that this expansion, rewritten in terms of u gives the *large u* expansion of \mathbf{q}_1 in the scale 3. So the first line (originating from the leading order in g in (3.5)) gives the leading large u term to all orders in g in this scale, the second line in (3.6) gives the subleading in large u term to all orders in g etc. This information is essential for the correct identification of \mathbf{q}_1 in the scale 3.

Now let us finally describe the situation in the scale 3. In this scale the 4th order finite difference equation cannot be much simplified but it can be solved iteratively order by order in the coupling g using the highly universal procedure from [20]. For instance at the first two orders we start by finding 4 independent solutions for $q = \mathbf{q} e^{\pm \pi u}$,

$$\begin{aligned} q_I &= 1 + g^2 \left(4iu \eta_2 \cos^2 \frac{\theta}{2} + 2 \eta_1 \cot^2 \frac{\theta}{2} ((u+i) \cos \theta + u - i) \right. \\ &\quad \left. + \frac{\cot^2 \frac{\theta}{2} (2u^3 \cos \theta + 2u^3 - 2u - i)}{u} \right), \\ q_{II} &= u, \\ q_{III} &= u^2, \\ q_{IV} &= 4\eta_1 u \cos^2 \frac{\theta}{2} - \frac{i}{u}. \end{aligned} \quad (3.7)$$

However, to be able to use the key analyticity condition (2.12) we need to identify \mathbf{q}_1 (or \mathbf{q}_2). That is, we have to find a linear combination of q_I, \dots, q_{IV} which matches (3.6) at large u . From this condition one finds uniquely

$$\mathbf{q}_1 = e^{\pi u} (A_I q_I + A_{II} q_{II} + A_{III} q_{III} + A_{IV} q_{IV}) + \mathcal{O}(g^4) \quad (3.8)$$

where

$$A_I = 1 + \hat{g}^2 \left(4 \log(2\Omega) + 2 \csc^2 \frac{\theta}{2} + i \frac{\Omega}{\hat{g}^2} + 2\pi i - 4 + 8\gamma \right) \quad (3.9)$$

$$A_{II} = 0 + \Omega(2 \log(2\Omega) + i\pi + 4\gamma - 2) \quad (3.10)$$

$$A_{III} = 0 - \hat{g}^2 \left(4 \cot^2 \frac{\theta}{2} \right) \quad (3.11)$$

$$A_{IV} = 0 - \hat{g}^2 \left(\csc^2 \frac{\theta}{2} + \frac{i\Omega}{2\hat{g}^2} \sec^2 \frac{\theta}{2} \right). \quad (3.12)$$

In this way we deduce \mathbf{q}_1 . This allows us to find $\tilde{\mathbf{q}}_1(u) = \mathbf{q}_1(-u)$ via (2.12). On the last step of the procedure we consider the combinations

$$\mathbf{q}_1(u) + \tilde{\mathbf{q}}_1(u), \quad \frac{\mathbf{q}_1(u) - \tilde{\mathbf{q}}_1(u)}{\sqrt{u^2 - 4g^2}} \quad (3.13)$$

in which the cut on the real axis disappears. As at weak coupling the cuts manifest themselves as poles, thus the poles at the origin which are naturally present in \mathbf{q}_1 should cancel in these combinations [14]. This condition fixes the coefficients A_n and B_n and also the value of the energy at the given order in g . So, for instance, at the g^2 order we find the following expansion at the origin

$$\mathbf{q}_1(u) \simeq [1 + \pi u + \mathcal{O}(u^2)] - \Omega \left[-\frac{\sec^2 \frac{\theta}{2}}{2u} + \mathcal{O}(u^0) \right] + \mathcal{O}(g^4). \quad (3.14)$$

Then regularity of the second combination in (3.13) relates the singular term proportional to Ω with the linear coefficient πu so that we get

$$\Omega = 4\pi g^2 \cos^2 \frac{\theta}{2} + \mathcal{O}(g^4) . \quad (3.15)$$

This perfectly matches the well known leading order result.

3.2 Expansion to high order in the coupling

The procedure described above allows to efficiently generate the quark-anti-quark potential expanded to very high orders in g . We have computed the expansion up to g^{14} order. The result up to g^{10} order is shown below

$$\begin{aligned} \frac{\Omega}{4\pi} = & \hat{g}^2 + \quad (3.16) \\ & \hat{g}^4 [16L - 8] + \\ & \hat{g}^6 \left[128L^2 + L \left(64 + \frac{64\pi^2 T}{3} \right) - 112 - \frac{8\pi^2}{3} + 72T\zeta_3 \right] + \\ & \hat{g}^8 \left[\frac{2048L^3}{3} + \frac{1024}{3}\pi^2 L^2 T + 2048L^2 + LT \left(768\zeta_3 + \frac{2176\pi^2}{3} \right) + \left(-768 - \frac{640\pi^2}{3} \right) L \right. \\ & \quad \left. + T^2 (128\pi^2\zeta_3 - 760\zeta_5) + T \left(384\zeta_3 - 640\pi^2 + \frac{32\pi^4}{9} \right) + \frac{1664\zeta_3}{3} + \frac{1216\pi^2}{9} - 1280 \right] + \\ & \hat{g}^{10} \left[\frac{8192L^4}{3} + \frac{8192}{3}\pi^2 L^3 T + \frac{57344L^3}{3} + \frac{2048}{9}\pi^4 L^2 T^2 + L^2 T \left(3072\zeta_3 + \frac{71680\pi^2}{3} \right) \right. \\ & \quad \left. + \left(20480 - \frac{19456\pi^2}{3} \right) L^2 + LT^2 \left(\frac{8704\pi^2\zeta_3}{3} - 6400\zeta_5 + \frac{2560\pi^4}{3} \right) \right. \\ & \quad \left. + LT \left(12800\zeta_3 - \frac{46592\pi^2}{3} - \frac{6656\pi^4}{45} \right) + L \left(\frac{26624\zeta_3}{3} - 26624 + \frac{38912\pi^2}{9} \right) \right. \\ & \quad \left. + T^3 \left(\frac{1792\pi^4\zeta_3}{45} - \frac{4928\pi^2\zeta_5}{3} + 8624\zeta_7 \right) \right. \\ & \quad \left. + T^2 \left(3392\pi^2\zeta_3 + 1248\zeta_3^2 - 4000\zeta_5 - \frac{1024\pi^4}{3} - \frac{16\pi^6}{45} \right) \right. \\ & \quad \left. + T \left(896\zeta_3 + \frac{3392\pi^2\zeta_3}{3} + 1600\zeta_5 - \frac{10112\pi^2}{3} + \frac{1408\pi^4}{45} \right) \right. \\ & \quad \left. + 6656\zeta_3 + \frac{736\pi^4}{45} + \frac{5824\pi^2}{27} - \frac{37888}{3} \right] . \end{aligned}$$

Here we use the following notation

$$\hat{g} \equiv g \cos \frac{\theta}{2} , \quad T \equiv \frac{1}{\cos^2 \frac{\theta}{2}} , \quad L \equiv \log \sqrt{8e^\gamma \pi \hat{g}^2} . \quad (3.17)$$

In appendix C we also give the expression for the quite lengthy \hat{g}^{12} and \hat{g}^{14} orders. They are particularly interesting since at order \hat{g}^{12} an irreducible multiple zeta value appears for the first time (namely, $\zeta_{6,2}$). We also give the full 7-loop result in a Mathematica notebook attached to this paper.

We notice that at the g^{2n+2} order the result is a n th order polynomial in L and T . The terms with the maximal power of L and the subleading in L terms have a very simple structure which can be summarized by the following formula

$$\frac{\Omega}{4\pi} = \sum_{n=0}^{\infty} \hat{g}^{2n+2} \frac{16^n L^n}{n!} \left(1 + \frac{3n^2 - 5n}{4L} + \pi^2 T \frac{n^2 - n}{12L} + \mathcal{O}(1/L^2) \right). \quad (3.18)$$

Our 7-loop result computed from the QSC is in perfect agreement with direct field theory perturbative calculations. The first three orders were known completely and were computed in [1, 3–8]. In addition, our formula (3.18) matches the all-orders prediction of [3] for the coefficients of the leading logarithmic terms $\hat{g}^{2n} \log^{n-1} \hat{g}$. We also reproduced⁴ the result of [6] for the subleading logarithmic term at 4th nontrivial order (i.e. $\hat{g}^8 \log^2 \hat{g}$).

In the next section we will show that the terms which do not contain T can be captured by a much simpler set of equations.

4 Ladders limit of the quark-anti-quark potential

A remarkable special limit, revealing rich structures, is the “double scaling” limit when the twist $t = e^{i\theta/2}$ scales to zero as g . In this limit the effective coupling $\hat{g} = \frac{g}{2t}(1 + t^2)$ and $\Omega(\hat{g})$ remain finite. It is expected that in this special case our system of equations can be solved exactly to all orders in \hat{g} or at least simplified considerably. From the gauge theory side, only the ladder diagrams contribute in that limit. Their resummation is achieved by Bethe-Salpeter techniques which results in a Schrödinger equation [1, 4]

$$F''(z) + F(z) \left(\frac{4\hat{g}^2}{z^2 + 1} - \frac{\Omega^2}{4} \right) = 0, \quad (4.1)$$

whose ground state energy gives the quark-anti-quark potential $\Omega(\hat{g})$. Its expansion in small \hat{g} should capture all terms in (3.16) without T to all orders in \hat{g} , as $T \rightarrow 0$ in this limit. Below we will demonstrate how this Schrödinger equation is encoded into the QSC.

4.1 Double scaling limit of the QSC

The main simplification in this limit occurs because $g \rightarrow 0$ and thus each of the cuts $[-2g, 2g]$ collapses into a point. In particular this implies that $\mathbf{f}(u)$ and $\mathbf{g}(u)$ from (2.1), as analytic functions everywhere except the cut, reduce to simple rational functions. Nevertheless, the result is a nontrivial function of the coupling \hat{g} which resums the usual perturbative expansion. In this sense this setup reminds the BFKL limit of the QSC studied in [20, 26]. Special care should be taken with the exponents $e^{\pm\theta u}$ in \mathbf{P}_a which give extra factors of t or $1/t$ each time we shift the argument u by $\pm i/2$. For this reason we have to keep terms up to order t^4 in \mathbf{P}_a . Assuming all the coefficients $A_n, B_n \sim 1$ (which we initially deduced from the weak coupling solution described in section 3, and confirmed

⁴Some of the perturbative field theory calculations discussed here were done for the special case $\theta = 0$ only.

by self-consistency) we get

$$\begin{aligned} \mathbf{f}(u) &= \frac{1}{u} + \frac{4\hat{g}^2 t^2 (A_1 u + 1)}{u^3} + \frac{8\hat{g}^2 t^4 (2\hat{g}^2 (A_2 u^2 + 2A_1 u + 2) - u^2 (A_1 u + 1))}{u^5} + \mathcal{O}(t^6) \\ \mathbf{g}(u) &= u - i + t^2 \left[\frac{4\hat{g}^2 (A_1 u^2 + B_1 + u - i)}{u^2} + 4i \right] + t^4 \left[-\frac{8\hat{g}^2 (A_1 u^2 + B_1 + u - 3i)}{u^2} \right. \\ &\quad \left. + \frac{16\hat{g}^4 (A_1 u^2 + (B_2 + 2)u + 2(B_1 - i))}{u^4} - 2i \right] + \mathcal{O}(t^6) \end{aligned} \quad (4.2)$$

We can also exclude B_1 using the expression for Ω (2.23),

$$B_1 = 2i + t^2 \left(4A_1 \hat{g}^2 + 4iA_2 \hat{g}^2 - \frac{i\Omega^2}{\hat{g}^2} - 4i \right) + \mathcal{O}(t^4). \quad (4.3)$$

Next we plug the expressions (4.2) into (2.9) and expand to the leading order in t . We notice that the dependence on all remaining A_n and B_n disappears and we simply get

$$\begin{aligned} &\left(\frac{16\hat{g}^4}{u^3} + \frac{16\hat{g}^2}{u} - \frac{4\Omega^2}{u} + 6u \right) q(u) \\ &+ (u+i)q(u+2i) - \left(\frac{4\hat{g}^2(2u+i)}{u(u+i)} + 4u+2i \right) q(u+i) \\ &+ (u-i)q(u-2i) - \left(\frac{4\hat{g}^2(2u-i)}{u(u-i)} + 4u-2i \right) q(u-i) = 0 \end{aligned} \quad (4.4)$$

where $q(u) = \mathbf{Q}(u)e^{\pm\pi u}/\sqrt{u}$. A great simplification comes from the fact that this equation can be factorized into two second order equations! This allows to replace (4.4) by a pair of second order equations

$$-2q(u) (2\hat{g}^2 - \Omega u + u^2) + u^2 q(u-i) + u^2 q(u+i) = 0 \quad (4.5)$$

and the second one related by $\Omega \rightarrow -\Omega$. By analyzing the large u asymptotics it is easy to see that the two solutions of (4.5) correspond to \mathbf{Q}_1 and \mathbf{Q}_4 . To fix the conventions and normalizations we define

$$q_1 \simeq \sqrt{\pi/2}^{1/4} \sqrt{8\Omega u} e^{-\sqrt{8\Omega u}}, \quad q_4 \simeq \frac{1}{16i\pi\Omega^2 t^4} \sqrt{\pi/2}^{1/4} \sqrt{8\Omega u} e^{+\sqrt{8\Omega u}}, \quad q_4(0) = 0 \quad (4.6)$$

where

$$q_1 = e^{-\pi u} \mathbf{Q}_1 / \sqrt{u}, \quad q_4 = e^{+\pi u} \mathbf{Q}_4 / \sqrt{u}. \quad (4.7)$$

The relative coefficient in (4.6) is chosen in agreement with the canonical normalization (2.28). We also choose q_1 and q_4 to be regular in the upper half plane as usual. We see that (4.5) is invariant under complex conjugation, which implies that \bar{q}_1 and \bar{q}_4 are some linear combinations of q_1 and q_4 with i -periodic coefficients

$$\bar{q}_1 = \Omega_1^1 q_1 + e^{-2\pi u} \Omega_1^4 q_4 \quad (4.8)$$

$$\bar{q}_4 = e^{+2\pi u} \Omega_4^1 q_1 + \Omega_4^4 q_4. \quad (4.9)$$

Here Ω_i^j are some i -periodic functions for which notation is introduced in accordance with the general consideration from appendix D. Knowing the analytical properties of q_1 and q_4 , which follow from the equation (4.5), we can constrain the possible form of Ω_i^j . From the equation (4.5) we can see that q_1 should have double poles at $u = -2in$ for $n = 1, 2, \dots$ due to the u^2 factors in the equation. Similarly q_4 has simple poles at the same points due to the additional condition $q_4(0) = 0$ which softens the singularity. Furthermore, the complex conjugate functions \bar{q}_1 and \bar{q}_4 should have the same poles as q_1 and q_4 but in the upper half-plane instead of the lower half-plane. The poles of \bar{q}_1 in the upper half plane can only originate from Ω' s in the r.h.s. of (4.8). This implies that Ω_1^1 and Ω_1^4 can have at most 2nd order poles, similarly, Ω_4^1 and Ω_4^4 can only have simple poles. Next, if we expand (4.8) near $u = 0$ in order to cancel poles in the r.h.s. we must assume that Ω_1^1 has simple pole only as $\bar{q}_4(0) = 0$. Similarly Ω_4^1 should be regular. Finally, since for large u the asymptotics of q_1 does not contain periodic exponents due to the definition (4.6) we can write the following ansatz for Ω' s in terms of a few constants a_i :

$$\Omega_1^1 = \frac{a_1 + a_2 e^{2\pi u}}{e^{2\pi u} - 1}, \quad \Omega_1^4 = \frac{a_3 e^{2\pi u} + a_0}{(e^{2\pi u} - 1)^2}, \quad \Omega_4^1 = a_4 e^{-2\pi u}, \quad \Omega_4^4 = \frac{a_5 + a_6 e^{2\pi u}}{e^{2\pi u} - 1}. \quad (4.10)$$

We also note that $a_0 = 0$ since Ω_1^4 should be even as explained in (D.7). By comparing the large u asymptotics in the first equation of (4.8) at $u \rightarrow -\infty$ we can fix a_3 and get Ω_1^4

$$a_3 = 16\pi t^4 \Omega^2, \quad \Omega_1^4 = \frac{4\pi t^4 \Omega^2}{\sinh^2(\pi u)}. \quad (4.11)$$

This allows to close the equations. Indeed, by rewriting

$$\Omega_1^4 = \left[\frac{\tilde{\Omega}_1^4 + \Omega_1^4}{2} \right] - \left[\frac{\tilde{\Omega}_1^4 - \Omega_1^4}{2\sqrt{u^2 - 4g^2}} \right] \sqrt{u^2 - 4g^2} \quad (4.12)$$

so that the expressions in the square brackets are regular at the origin to all orders in g we see that the poles present in (4.11) can only originate from the last term. At the same time the last term can be written in terms of q and \bar{q} using (D.7):

$$\left[\frac{\tilde{\Omega}_1^4 - \Omega_1^4}{2\sqrt{u^2 - 4g^2}} \right] = -\frac{u\bar{q}_1(-u)q_1(-u)e^{-2\pi u} - u\bar{q}_1(u)q_1(u)e^{+2\pi u}}{2u} + \mathcal{O}(g^2) = bu + \mathcal{O}(u^3) + \mathcal{O}(g^2) \quad (4.13)$$

which results in the following pattern of the leading singularities in Ω_1^4

$$\Omega_1^4 = \frac{2bg^4}{u^2} + \frac{4bg^6}{u^4} + \dots + \text{less singular terms} \quad (4.14)$$

thus we can relate b to $\Omega(\hat{g})$ as

$$b = \frac{\Omega^2(\hat{g})}{8\pi\hat{g}^4} \quad (4.15)$$

or

$$\frac{\Omega^2(\hat{g})}{8\pi\hat{g}^4} = \lim_{u \rightarrow 0} \frac{\bar{q}_1(u)q_1(u)e^{+2\pi u} - \bar{q}_1(0)q_1(0)}{u}. \quad (4.16)$$

This condition together with the finite difference equation (4.5) allows to determine $\Omega(\hat{g})$. Namely, we have to find such value of the parameter Ω in the finite difference equation (4.5) for which its solution q_1 with the asymptotic (4.6), expanded at the origin, satisfies the condition (4.16). This type of problem can be easily solved numerically or perturbatively in \hat{g} .

To solve the system perturbatively we repeat basically the same steps as in the previous section, with an additional simplification that we do not have to tune any parameters in \mathbf{P}_a except $\Omega(\hat{g})$, and that we only have to deal with the second order equation instead of the 4th order equation. This procedure, explained in detail in section 3.1, leads to the following result

$$\begin{aligned}
 \frac{\Omega(\theta = i\infty)}{4\pi} = & \hat{g}^2 + \\
 & \hat{g}^4 [16L - 8] + \\
 & \hat{g}^6 \left[128L^2 + 64L - 112 - \frac{8\pi^2}{3} \right] + \\
 & \hat{g}^8 \left[\frac{2048L^3}{3} + 2048L^2 - \left(768 + \frac{640}{3}\pi^2 \right) L - 1280 + \frac{1216}{9}\pi^2 + \frac{1664}{3}\zeta_3 \right] + \\
 & \hat{g}^{10} \left[\frac{8192L^4}{3} + \frac{57344L^3}{3} + \left(20480 - \frac{19456\pi^2}{3} \right) L^2 - \left(26624 - \frac{38912\pi^2}{9} - \frac{26624\zeta_3}{3} \right) L \right. \\
 & \quad \left. - \frac{37888}{3} + \frac{5824\pi^2}{27} + 6656\zeta_3 + \frac{736\pi^4}{45} \right] + \\
 & \hat{g}^{12} \left[\frac{131072L^5}{15} + \frac{327680L^4}{3} + \left(\frac{1048576}{3} - \frac{1097728\pi^2}{9} \right) L^3 \right. \\
 & \quad \left. + L^2 \left(\frac{212992\zeta_3}{3} + 81920 + 24576\pi^2 \right) \right. \\
 & \quad \left. + L \left(212992\zeta_3 - \frac{1515520}{3} + \frac{1776640\pi^2}{27} + \frac{39424\pi^4}{15} \right) \right. \\
 & \quad \left. + \frac{124928\zeta_5}{5} + \frac{106496\zeta_3}{3} + \pi^2 \left(-\frac{93184\zeta_3}{9} - \frac{107008}{27} \right) - \frac{1159424\pi^4}{675} - \frac{295936}{3} \right] + \\
 & \hat{g}^{14} \left[\frac{1048576L^6}{45} + \frac{6815744L^5}{15} + \left(2752512 - \frac{15007744\pi^2}{9} \right) L^4 \right. \\
 & \quad \left. + L^3 \left(\frac{3407872\zeta_3}{9} + \frac{15073280}{3} - \frac{11141120\pi^2}{9} \right) \right. \\
 & \quad \left. + L^2 \left(2555904\zeta_3 - \frac{6914048}{3} + \frac{17096704\pi^2}{9} + \frac{7221248\pi^4}{45} \right) \right. \\
 & \quad \left. + L \left(\pi^2 \left(\frac{61534208}{81} - \frac{5324800\zeta_3}{9} \right) + \frac{9797632\zeta_3}{3} + \frac{1998848\zeta_5}{5} - \frac{23560192}{3} - \frac{34199552\pi^4}{225} \right) \right. \\
 & \quad \left. + 499712\zeta_5 + \pi^2 \left(106496\zeta_3 - \frac{67858432}{243} \right) + \frac{1384448\zeta_3^2}{9} - \frac{1384448\zeta_3}{3} - \frac{122624\pi^6}{945} \right. \\
 & \quad \left. + \frac{274300928\pi^4}{10125} - \frac{4759552}{15} \right]
 \end{aligned} \tag{4.17}$$

where as before $L \equiv \log \sqrt{8e\gamma\pi\hat{g}^2}$. We notice that all the terms in (3.16) without T are reproduced perfectly by the above expansion.

In the next section we will show how to rewrite this finite difference ‘boundary’ problem into a spectral problem of a Schrödinger equation by performing a kind of Mellin transformation.

4.2 Equivalence to the Schrödinger equation

The double scaling limit of the quark-anti-quark potential has a long history. In [1, 4] it was shown that in this limit only the ladder diagrams contribute and they can be resummed by a Bethe-Salpeter equation. This problem can be reformulated as a problem of finding the ground-state energy of the Schrödinger equation

$$F''(z) + F(z) \left(\frac{4\hat{g}^2}{z^2 + 1} - \frac{\Omega^2}{4} \right) = 0. \quad (4.18)$$

The Schrödinger wavefunction is linked to the solution of the Bethe-Salpeter equation. In this section we will show that this problem is equivalent to the second order finite difference equation arising from the QSC accompanied by the “quantization condition” at the origin (4.16).

Relating q -function to the wave function. First we relate the q -function q_1 with the solution of (4.18) decaying at $+\infty$. We assume that the solution decaying at $+\infty$ is normalized so that

$$F(z) \simeq e^{-\Omega z/2}. \quad (4.19)$$

Let us show that the solution q_1 of (4.5) is given by the following integral Mellin-like transformation

$$\frac{q_1(u)}{u} = 2 \int_i^{+\infty} \frac{e^{-\frac{\Omega z}{2}}}{z^2 + 1} \left(\frac{z+i}{z-i} \right)^{iu} F(z) dz, \quad \text{Im } u > 0. \quad (4.20)$$

To see that the equation (4.5) is indeed satisfied we consider an integral of a total derivative:

$$2 \int_i^{\infty} \partial_z \left(\left[(z^2 + 1)F'(z) + \frac{1}{2}F(z)(-4u + \Omega + \Omega z^2) \right] \frac{e^{-\frac{\Omega z}{2}}}{z^2 + 1} \left(\frac{z+i}{z-i} \right)^{iu} \right) dz \quad (4.21)$$

the boundary terms vanish for $\text{Im } u > 1$ and the integral is zero. At the same time evaluating the derivative and excluding the second derivative $F''(u)$ using (4.18) we get

$$\begin{aligned} 0 &= 2 \int_i^{\infty} \left[(-4\hat{g}^2 + 2\Omega u - 2u^2) + u(u+i) \left(\frac{z-i}{z+i} \right) + u(u-i) \left(\frac{z+i}{z-i} \right) \right] \frac{F(z)e^{-\frac{\Omega z}{2}}}{z^2 + 1} \left(\frac{z+i}{z-i} \right)^{iu} dz \\ &= (-4\hat{g}^2 + 2\Omega u - 2u^2) \frac{q_1(u)}{u} + u(u+i) \frac{q_1(u+i)}{u+i} + u(u-i) \frac{q_1(u-i)}{u-i}, \end{aligned} \quad (4.22)$$

which shows that $q_1(u)$ defined by the integral (4.20) satisfies (4.5). At the same time it is easy to see by the saddle-point analysis that $F(z) \simeq e^{-\Omega z/2}$ implies the following large u asymptotics for q_1 :

$$q_1(u) \simeq \sqrt{\pi/2}^{1/4} \sqrt{8\Omega u} e^{-\sqrt{8\Omega u}}. \quad (4.23)$$

Note that this map from $F(z)$ to $q_1(u)$ is valid for any (positive) value of Ω . Clearly, we have to additionally impose the decay of $F(z)$ at $z \rightarrow -\infty$ to constrain Ω . At the same time from the QSC point of view we should impose on q_1 the condition (4.16) at the origin. Below we show that these two conditions are equivalent.

Equivalence of the two quantization conditions. We should relate the behavior of $q_1(u)$ near the origin with the normalizability of $F(z)$ as a solution of the Schrödinger equation. It is clear that the singularity in $q(u)/u$ around $u = 0$ is due to the divergence in the integral (4.20) near $z = i$. Therefore it is controlled by the behavior of $F(z)$ at $z = i$. So our problem seems to be rather nontrivial as we have to relate the values of F at large z with its behavior near $z = i$. In general that would be impossible to do without an explicit solution. However, we noticed an interesting duality of the equation which allows to do this.

The key observation is that for the normalizable $F(z)$ its Fourier image satisfies essentially the same differential equation. More precisely, defining $G(k)$ as

$$\frac{G(k)}{k^2 + 1} = \frac{\Omega^{3/2}}{8\hat{g}\sqrt{\pi}} \int_{-\infty}^{\infty} dz F(z) e^{ik\frac{\Omega}{2}z} \quad (4.24)$$

it is easy to see that $G(k)$ satisfies literally the same Schrödinger equation (4.18). Furthermore, $G(k)$ also decays exponentially at both infinities as Fourier transform of a smooth function and is also smooth since F itself decays exponentially at both infinities. This means that F and G should in fact coincide up to a constant factor. To make the symmetry more manifest we can write the relation (4.24) between F and G as

$$F(z) = \frac{2\hat{g}}{\sqrt{\pi\Omega}} \int_{-\infty}^{\infty} dk \frac{G(k)}{k^2 + 1} e^{ik\frac{\Omega}{2}z}, \quad G(k) = \frac{2\hat{g}}{\sqrt{\pi\Omega}} \int_{-\infty}^{\infty} dz \frac{F(z)}{z^2 + 1} e^{ik\frac{\Omega}{2}z}. \quad (4.25)$$

We see that in the normalization (4.24)⁵ we must have $G(z) = F(z)$, so that we get

$$F(k) = \frac{2\hat{g}}{\sqrt{\pi\Omega}} \int_{-\infty}^{\infty} dz \frac{F(z)}{z^2 + 1} e^{ik\frac{\Omega}{2}z}. \quad (4.26)$$

This property of the solution $F(z)$ allows to bootstrap the behavior at infinity and near the branch point $z = i$. Let's assume that $F(z)$ has the following expansion near $z = i$:

$$F(z) = -\frac{iC}{2\hat{g}^2} + C(z - i) \log(iz + 1) + \dots \quad (4.27)$$

which is obtained by solving the equation (4.18) in the vicinity of $z = i$. As $z = i$ is the closest to the real axis singularity of $F(z)$ it controls the large z behavior of $F(z)$

$$F(k) \simeq \frac{2\hat{g}}{\sqrt{\pi\Omega}} \int_{-\infty}^{\infty} dw \frac{-\frac{iC}{2\hat{g}^2}}{w^2 + 1} e^{ik\frac{\Omega}{2}w} = \frac{-iC}{\hat{g}} \sqrt{\frac{\pi}{\Omega}} e^{-\frac{k\Omega}{2}} \quad (4.28)$$

⁵There is a possibility that $G(z) = -F(z)$, however, it is easy to see that since $F(z) > 0$ for real z so must be G .

next using the normalization (4.19) we find

$$C = i \frac{\hat{g} \sqrt{\Omega}}{\sqrt{\pi}}, \quad (4.29)$$

which fixes the expansion (4.27) near $z = i$. This allows to find the residue of $q_1(u)/u$ at the origin by plugging (4.27) into (4.20):

$$\frac{q_1(u)}{u} \simeq \frac{i C e^{-\frac{i\Omega}{2}}}{2\hat{g}^2 u} = -\frac{1}{u} \frac{e^{-\frac{i\Omega}{2}} \sqrt{\Omega}}{2\hat{g} \sqrt{\pi}}. \quad (4.30)$$

In appendix E we describe how to use a similar technique to establish the subleading coefficient in u which then gives:

$$e^{2\pi u} q_1(u) \bar{q}_1(u) = -\frac{C^2}{4\hat{g}^4} - \frac{C^2 \Omega}{8\hat{g}^6} u + \mathcal{O}(u^2) = \frac{\Omega}{4\pi \hat{g}^2} + \frac{u \Omega^2}{8\pi \hat{g}^4} + \mathcal{O}(u^2) \quad (4.31)$$

showing that the condition (4.16), coming from the depth of QSC, does hold! This finishes the proof of equivalence between the QSC and the Schrödinger equation in the ladders limit.

5 Numerical solution in a wide range of the coupling

The QSC can be very efficiently solved numerically with essentially arbitrary precision at finite values of the coupling and all other parameters. The general method, which is also applicable here, was developed in [27]. We have used it to generate numerical values for the quark-anti-quark potential in a wide range of the 't Hooft coupling with ~ 20 digits precision. Our method works well for arbitrary real θ , but we decided to focus on the case $\theta = 0$. Our numerical data is listed in appendix F. A plot of our results is shown on figure 1.

Let us make a comparison with the known analytical predictions. At strong coupling the classical [9, 10] and 1-loop [11, 12] string theory results read

$$\Omega \simeq \frac{\pi(4\pi g + a_1)}{4K\left(\frac{1}{2}\right)^2} = 2.8710800442g - 0.3049193809. \quad (5.1)$$

At the same time a fit of our numerical data gives

$$\Omega = 2.8710800436g - 0.3049193819 + \frac{0.0100740}{g} + \frac{0.000381}{g^2} + \dots \quad (5.2)$$

which quite convincingly reproduces the first two known orders.

At weak coupling one can see on the plot that this expansion matches well our numerics. In addition, our analytic solution of the QSC at weak coupling described in section 3 provides the expansion of Ω to first 7 loop orders presented in (3.16) and in appendix C. Fixing a particular small value of the coupling $g = 0.0625$ we compared our numerical prediction $\Omega = 0.04472043670132964806$ at this point with the analytic weak coupling expansion. In table 1 one clearly sees that including more and more orders in the expansion improves noticeably the agreement with our numerical result. This is a nice check of our weak coupling analytic prediction.

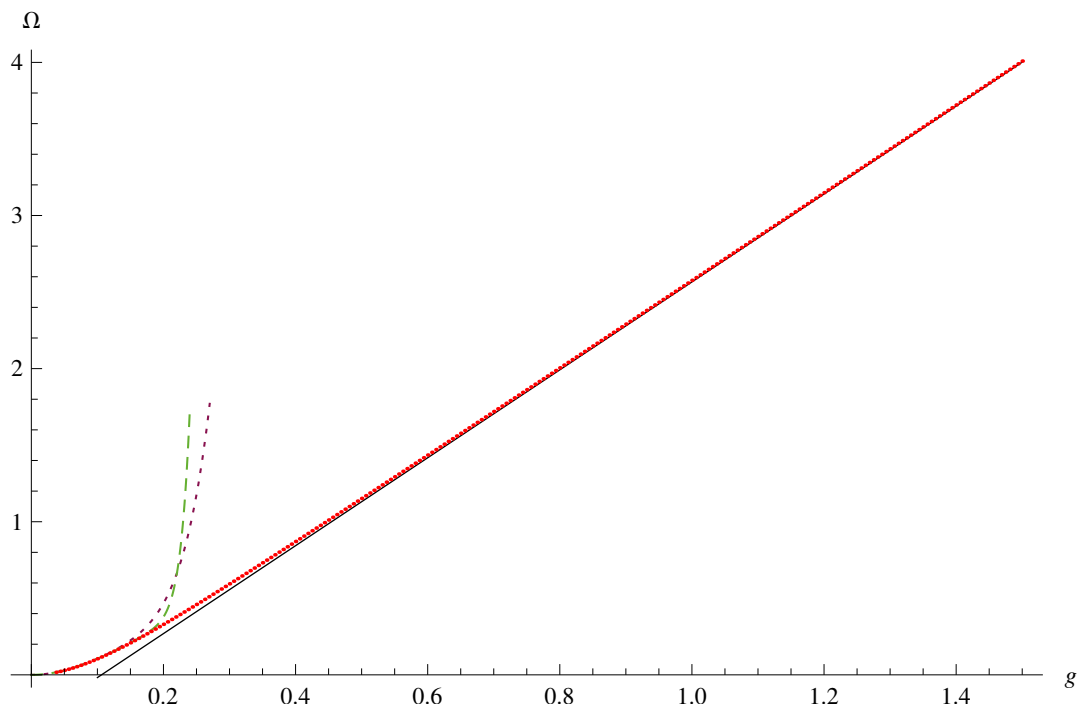


Figure 1. Numerical results for the quark-anti-quark potential $\Omega(g)$ at $\theta = 0$. Our numerical data points are shown in red, while the solid black line shows the strong coupling analytic prediction (5.1). The purple curve is the 3-loop weak coupling expansion, and the dashed green curve is our 7-loop perturbative result.

| | $\Omega^{\text{perturbative}}$ | $\Omega^{\text{numerical}}$ | difference |
|--------|--------------------------------|-----------------------------|---------------|
| 1-loop | 0.04908738521 | 0.04472043670 | 0.00436694851 |
| 2-loop | 0.04487846353 | 0.04472043670 | 0.00015802682 |
| 3-loop | 0.04473327069 | 0.04472043670 | 0.00001283399 |
| 4-loop | 0.04471883557 | 0.04472043670 | 0.00000160113 |
| 5-loop | 0.04472038490 | 0.04472043670 | 0.00000005179 |
| 6-loop | 0.04472043227 | 0.04472043670 | 0.00000000442 |
| 7-loop | 0.04472043747 | 0.04472043670 | 0.00000000076 |

Table 1. Comparison between the 7-loop weak coupling prediction and the numerical data for the quark-anti-quark potential at $g = 0.0625$.

6 Conclusion

In this paper we demonstrated that the Quantum Spectral Curve approach allows to deeply explore the quark-anti-quark potential in a variety of settings. In particular, we generated highly precise numerical data at finite coupling interpolating extremely well between gauge theory and string theory predictions. Thus finally we are able to access on a fully nonperturbative level this observable which historically has been a milestone in the investigations of AdS/CFT.

The setup we study corresponds to a singular limit $\phi \rightarrow \pi$ of the cusp anomalous dimension which leads to a drastic change of Q-functions' asymptotics in the QSC. The asymptotics we found are of a novel type even for integrable systems with twisted boundary conditions. As this is yet another set of nontrivial asymptotics in the QSC, it is clearly an important question how to classify all possible types of asymptotics. They should correspond to some kind of deformations and boundary problems for local or nonlocal observables likely including the setups studied in [28, 29]. Consistency of asymptotics with the functional QSC equations appears to be a highly nontrivial constraint giving hope for an exhaustive description.

Using the efficient iterative procedure of [20] we computed the weak coupling expansion of the potential to the 7th loop order. The perturbative expansion is known to be rather nontrivial and to be captured by an effective theory arising at low energy scales. Remarkably, we also observed the appearance of several distinct scales in the QSC which may be thought of as a counterpart to this effective field theory description. In the future it will be also interesting to apply the QSC to study the energies of hydrogen-like bound states in $\mathcal{N} = 4$ SYM [30] which are also related to a $\phi \rightarrow \pi$ limit. Moreover, our weak coupling results may be useful to establish connections with QCD, similarly to e.g. [31] (see also e.g. [32–34] for some details of the QCD calculations of the quark-antiquark potential).

We also studied the double scaling limit when the twist θ in the scalar sector goes to $i\infty$. We showed how the Schrödinger equation arising on the field theory side from resummation of ladder diagrams is encoded in the QSC, with its wavefunction rather directly linked to the Q-functions. We believe that this approach should also apply to a similar double scaling limit of γ -deformed $\mathcal{N} = 4$ SYM recently proposed in [16], where the QSC has many common features with the one for the cusped Wilson lines setup [15, 35, 36].⁶ This limit in the γ -deformed model was advocated in [16] to give a novel integrable 4d theory.

We also observed a peculiar duality of the Schrödinger equation with respect to Fourier transform, whose meaning in the QSC itself beyond this special limit calls for further clarification and might have something to do with dual conformal symmetry. Viewing the relation between the QSC and the Schrödinger equation as a kind of ODE/IM correspondence [41], it would be interesting to see what kind of generalization will take place at finite twist. Another important direction is to derive the Schrödinger equation of [4] (see also [42]) in the ladders limit with generic ϕ .

Finally, as the ladders limit allows for a simpler access to the wrapping corrections, it could also serve as a useful ground to attempt a finite-size resummation of perturbation theory for 3-point correlators [43–47], using Q-functions as building blocks.

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⁶The Y-system and TBA for the spectrum in the γ -deformed case were proposed earlier in [37–40].

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A Weak coupling limit of the coefficients

At weak coupling one can fix the values of the several leading coefficients A_n, B_n which parameterize the \mathbf{P} -functions via (2.4). In order to do this we used the leading order weak coupling solution of the QSC constructed in [15]. With the \mathbf{P}_a and μ_{ab} functions from that paper one can build $\tilde{\mathbf{P}}_a = \mu_{ab} \chi^{bc} \mathbf{P}_c$ and compare the result with our ansatz (2.4) in which $\tilde{\mathbf{P}}_a$ is constructed by simply replacing $x \rightarrow 1/x$. For the case $\phi = \pi$ we found that

$$B_0 = \frac{\cos \theta - 2}{\sin \theta} + A_1 g^2 \quad (\text{A.1})$$

and the remaining coefficients to the leading order are all fixed as

$$A_n = \frac{2^{n-1} \pi^n (1 + (-1)^n)}{(n+1)!} + \mathcal{O}(g^2) \quad (\text{A.2})$$

$$B_{2n} = \frac{(2\pi)^{2n-2}}{(2n-1)!} + \mathcal{O}(g^2), \quad n > 1 \quad (\text{A.3})$$

$$B_{2n-1} = -\cot \theta \frac{(2\pi)^{2n-2}}{(2n-1)!} + \mathcal{O}(g^2), \quad n > 1 \quad (\text{A.4})$$

$$B_1 = 2 \tan \frac{\theta}{2} + \mathcal{O}(g^2) \quad (\text{A.5})$$

$$B_2 = 0 + \mathcal{O}(g^2). \quad (\text{A.6})$$

B Determinants entering the 5th order equation on \mathbf{Q}_i

The 4th order difference equation (2.9) on \mathbf{Q}_i includes several determinants built out of the \mathbf{P} -functions, which are defined as follows:

$$D_0 = \det \begin{pmatrix} \mathbf{P}^{1[+2]} & \mathbf{P}^{2[+2]} & \mathbf{P}^{3[+2]} & \mathbf{P}^{4[+2]} \\ \mathbf{P}^1 & \mathbf{P}^2 & \mathbf{P}^3 & \mathbf{P}^4 \\ \mathbf{P}^{1[-2]} & \mathbf{P}^{2[-2]} & \mathbf{P}^{3[-2]} & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \mathbf{P}^{2[-4]} & \mathbf{P}^{3[-4]} & \mathbf{P}^{4[-4]} \end{pmatrix}, \quad (\text{B.1})$$

$$D_1 = \det \begin{pmatrix} \mathbf{P}^{1[+4]} & \mathbf{P}^{2[+4]} & \mathbf{P}^{3[+4]} & \mathbf{P}^{4[+4]} \\ \mathbf{P}^1 & \mathbf{P}^2 & \mathbf{P}^3 & \mathbf{P}^4 \\ \mathbf{P}^{1[-2]} & \mathbf{P}^{2[-2]} & \mathbf{P}^{3[-2]} & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \mathbf{P}^{2[-4]} & \mathbf{P}^{3[-4]} & \mathbf{P}^{4[-4]} \end{pmatrix}, \quad (\text{B.2})$$

$$D_2 = \det \begin{pmatrix} \mathbf{P}^{1[+4]} & \mathbf{P}^{2[+4]} & \mathbf{P}^{3[+4]} & \mathbf{P}^{4[+4]} \\ \mathbf{P}^{1[+2]} & \mathbf{P}^{2[+2]} & \mathbf{P}^{3[+2]} & \mathbf{P}^{4[+2]} \\ \mathbf{P}^{1[-2]} & \mathbf{P}^{2[-2]} & \mathbf{P}^{3[-2]} & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \mathbf{P}^{2[-4]} & \mathbf{P}^{3[-4]} & \mathbf{P}^{4[-4]} \end{pmatrix}, \quad (\text{B.3})$$

$$\bar{D}_1 = \det \begin{pmatrix} \mathbf{P}^{1[-4]} & \mathbf{P}^{2[-4]} & \mathbf{P}^{3[-4]} & \mathbf{P}^{4[-4]} \\ \mathbf{P}^1 & \mathbf{P}^2 & \mathbf{P}^3 & \mathbf{P}^4 \\ \mathbf{P}^{1[+2]} & \mathbf{P}^{2[+2]} & \mathbf{P}^{3[+2]} & \mathbf{P}^{4[+2]} \\ \mathbf{P}^{1[+4]} & \mathbf{P}^{2[+4]} & \mathbf{P}^{3[+4]} & \mathbf{P}^{4[+4]} \end{pmatrix}, \quad (\text{B.4})$$

$$\bar{D}_0 = \det \begin{pmatrix} \mathbf{P}^{1[-2]} & \mathbf{P}^{2[-2]} & \mathbf{P}^{3[-2]} & \mathbf{P}^{4[-2]} \\ \mathbf{P}^1 & \mathbf{P}^2 & \mathbf{P}^3 & \mathbf{P}^4 \\ \mathbf{P}^{1[+2]} & \mathbf{P}^{2[+2]} & \mathbf{P}^{3[+2]} & \mathbf{P}^{4[+2]} \\ \mathbf{P}^{1[+4]} & \mathbf{P}^{2[+4]} & \mathbf{P}^{3[+4]} & \mathbf{P}^{4[+4]} \end{pmatrix}. \quad (\text{B.5})$$

C Six and seven loop results at weak coupling

Using the QSC we have computed the weak coupling expansion of the quark-antiquark potential at the first seven nontrivial orders. The first five orders are given in the main text in (3.16). Here we present the rather bulky 6- and 7-loop results.

6-loop result. The term of order \hat{g}^{12} in $\frac{\Omega}{4\pi}$ reads

$$\begin{aligned} & \frac{131072L^5}{15} + \frac{327680L^4}{3} + \frac{131072}{9}\pi^2L^4T + \frac{1048576L^3}{3} - \frac{1097728}{9}\pi^2L^3 \\ & + \frac{1163264}{3}\pi^2L^3T + \frac{32768}{9}\pi^4L^3T^2 + 81920L^2 + 24576\pi^2L^2 + \frac{212992\zeta_3L^2}{3} \\ & - \frac{8192}{3}\pi^2L^2T + 81920\zeta_3L^2T - \frac{77824}{5}\pi^4L^2T + \frac{475136}{9}\pi^4L^2T^2 \\ & + \left(\frac{65536}{3}\pi^2\zeta_3 - 5120\zeta_5\right)L^2T^2 + -\frac{1515520L}{3} + \frac{1776640\pi^2L}{27} + 212992\zeta_3L \\ & + \frac{39424\pi^4L}{15} - 251904\pi^2LT + 176128\zeta_3LT - \frac{16384}{27}\pi^4LT \\ & + \left(10240\zeta_5 - \frac{71680}{9}\pi^2\zeta_3\right)LT - \frac{118784}{9}\pi^4LT^2 + \left(\frac{573440}{3}\pi^2\zeta_3 - 99840\zeta_5\right)LT^2 \\ & + \left(3072\zeta_3^2 + \frac{70912}{405}\pi^6\right)LT^2 + \left(\frac{139264}{45}\pi^4\zeta_3 - 31232\pi^2\zeta_5 + 60928\zeta_7\right)LT^3 \\ & - \frac{295936}{3} - \frac{107008\pi^2}{27} + \frac{106496\zeta_3}{3} - \frac{1159424\pi^4}{675} + \left(\frac{124928\zeta_5}{5} - \frac{93184\pi^2\zeta_3}{9}\right) \\ & - \frac{1190528\pi^2T}{27} - 19456\zeta_3T + \frac{3045376\pi^4T}{405} + \left(\frac{212992}{3}\pi^2\zeta_3 + 27648\zeta_5\right)T \\ & + \left(1536\zeta_3^2 - \frac{14464\pi^6}{405}\right)T - \frac{50176}{3}\pi^4T^2 - \left(\frac{172288}{3}\pi^2\zeta_3 + 24320\zeta_5\right)T^2 \\ & + \left(18816\zeta_3^2 + \frac{17344\pi^6}{135}\right)T^2 + \left(\frac{72704}{45}\pi^4\zeta_3 + 19136\pi^2\zeta_5 - 38976\zeta_7\right)T^2 \\ & + \left(\frac{228352}{45}\pi^4\zeta_3 - \frac{107264}{3}\pi^2\zeta_5 + 43904\zeta_7\right)T^3 \\ & + \left(2496\zeta_{6,2} + \frac{20224}{3}\pi^2\zeta_3^2 - 31232\zeta_3\zeta_5 + \frac{55304\pi^8}{42525}\right)T^3 \\ & + \left(-\frac{2560}{3}\pi^4\zeta_5 + 21504\pi^2\zeta_7 - 102816\zeta_9\right)T^4 \end{aligned} \quad (\text{C.1})$$

At this order an irreducible multiple zeta value appears for the first time, given by $\zeta_{6,2} \simeq 0.017819740416836$.

7-loop result. The term of order \hat{g}^{14} in $\frac{\Omega}{4\pi}$ is given by

$$\begin{aligned}
 & \frac{1048576L^6}{45} + \frac{524288}{9}L^5\pi^2T + \frac{6815744L^5}{15} + \frac{262144}{9}L^4\pi^4T^2 - 65536L^4T\zeta_3 + \frac{40632320}{9}L^4\pi^2T \\
 & - \frac{15007744}{9}L^4\pi^2 + 2752512L^4 + \frac{131072}{81}L^3\pi^6T^3 + 65536L^3\pi^2T^2\zeta_3 + \frac{655360}{3}L^3T^2\zeta_5 \\
 & + \frac{12255232}{9}L^3\pi^4T^2 - \frac{64159744}{135}L^3\pi^4T - 65536L^3T\zeta_3 + \frac{13303808}{3}L^3\pi^2T + \frac{3407872L^3\zeta_3}{9} \\
 & - \frac{11141120}{9}L^3\pi^2 + \frac{15073280L^3}{3} + \frac{2080768}{45}L^2\pi^4T^3\zeta_3 - \frac{499712}{3}L^2\pi^2T^3\zeta_5 - 129024L^2T^3\zeta_7 \\
 & + 32768L^2\pi^6T^3 - \frac{2828288}{405}L^2\pi^6T^2 - 36864L^2T^2\zeta_3^2 + \frac{11444224}{3}L^2\pi^2T^2\zeta_3 + 20480L^2T^2\zeta_5 \\
 & + \frac{2351104}{3}L^2\pi^4T^2 - \frac{7610368}{9}L^2\pi^2T\zeta_3 - 40960L^2T\zeta_5 - \frac{27344896}{45}L^2\pi^4T + 1671168L^2T\zeta_3 \\
 & - 3817472L^2\pi^2T + \frac{7221248L^2\pi^4}{45} + 2555904L^2\zeta_3 + \frac{17096704L^2\pi^2}{9} - \frac{6914048L^2}{3} + \frac{8192}{9}L\pi^6T^4\zeta_3 \\
 & - \frac{133120}{3}L\pi^4T^4\zeta_5 + 369152L\pi^2T^4\zeta_7 - 628992LT^4\zeta_9 + \frac{1176832L\pi^8T^3}{42525} + \frac{210944}{3}L\pi^2T^3\zeta_3^2 \\
 & - 71680LT^3\zeta_3\zeta_5 + 30720LT^3\zeta_{6,2} + \frac{7872512}{15}L\pi^4T^3\zeta_3 - 1899520L\pi^2T^3\zeta_5 + 867328LT^3\zeta_7 \\
 & + \frac{212992}{27}L\pi^6T^3 - \frac{1150976}{15}L\pi^4T^2\zeta_3 + 665600L\pi^2T^2\zeta_5 - 268800LT^2\zeta_7 + \frac{2378752}{405}L\pi^6T^2 \\
 & + 43008LT^2\zeta_3^2 + \frac{757760}{3}L\pi^2T^2\zeta_3 - 1587200LT^2\zeta_5 - \frac{14838784}{9}L\pi^4T^2 - \frac{2152448L\pi^6T}{2835} \\
 & - 163840LT\zeta_3^2 + \frac{24051712}{9}L\pi^2T\zeta_3 + 364544LT\zeta_5 + \frac{390412288}{405}L\pi^4T + 2457600LT\zeta_3 \\
 & - \frac{39706624}{9}L\pi^2T - \frac{5324800}{9}L\pi^2\zeta_3 + \frac{1998848L\zeta_5}{5} - \frac{34199552L\pi^4}{225} + \frac{9797632L\zeta_3}{3} \\
 & + \frac{61534208L\pi^2}{81} - \frac{23560192L}{3} - \frac{11264}{105}\pi^6T^5\zeta_5 + \frac{73216}{5}\pi^4T^5\zeta_7 - 285120\pi^2T^5\zeta_9 \\
 & + 1271952T^5\zeta_{11} - \frac{10544\pi^{10}T^4}{93555} + \frac{91136}{9}\pi^4T^4\zeta_3^2 - \frac{520832}{3}\pi^2T^4\zeta_3\zeta_5 + 179424T^4\zeta_5^2 \\
 & + 361088T^4\zeta_3\zeta_7 + \frac{16768}{3}\pi^2T^4\zeta_{6,2} - 26432T^4\zeta_{8,2} + \frac{65536}{45}\pi^6T^4\zeta_3 - 63488\pi^4T^4\zeta_5 \\
 & + 401408\pi^2T^4\zeta_7 - 508032T^4\zeta_9 + \frac{5137792\pi^6T^3\zeta_3}{2835} - 768T^3\zeta_3^3 + 30976\pi^4T^3\zeta_5 \\
 & - \frac{941632}{3}\pi^2T^3\zeta_7 + \frac{2211904T^3\zeta_9}{3} - \frac{142816\pi^8T^3}{14175} + \frac{1183232}{3}\pi^2T^3\zeta_3^2 - 337664T^3\zeta_3\zeta_5 \\
 & + 17664T^3\zeta_{6,2} - \frac{256000}{3}\pi^4T^3\zeta_3 + \frac{1762304}{3}\pi^2T^3\zeta_5 + 367360T^3\zeta_7 - \frac{446464}{45}\pi^6T^3 \\
 & + \frac{2348512\pi^8T^2}{42525} - \frac{175360}{3}\pi^2T^2\zeta_3^2 + \frac{76288}{3}T^2\zeta_3\zeta_5 + 26880T^2\zeta_{6,2} + \frac{6986752}{45}\pi^4T^2\zeta_3 \\
 & + \frac{295424}{9}\pi^2T^2\zeta_5 - 611520T^2\zeta_7 - \frac{1111552}{405}\pi^6T^2 + 225792T^2\zeta_3^2 - \frac{2234624}{3}\pi^2T^2\zeta_3 \\
 & - 261120T^2\zeta_5 + \frac{3700736\pi^4T^2}{27} - \frac{2342656}{135}\pi^4T\zeta_3 + 131584\pi^2T\zeta_5 + 33152T\zeta_7 \\
 & + \frac{3462656\pi^6T}{2835} + 165888T\zeta_3^2 + \frac{3972608}{27}\pi^2T\zeta_3 + 387072T\zeta_5 - \frac{222660352\pi^4T}{1215} \\
 & - 544768T\zeta_3 + \frac{16618240\pi^2T}{81} - \frac{122624\pi^6}{945} + \frac{1384448\zeta_3^2}{9} + 106496\pi^2\zeta_3 + 499712\zeta_5 \\
 & + \frac{274300928\pi^4}{10125} - \frac{1384448\zeta_3}{3} - \frac{67858432\pi^2}{243} - \frac{4759552}{15}
 \end{aligned} \tag{C.2}$$

where we have a new multiple zeta value $\zeta_{8,2} \simeq 0.0041224696783998322240$.

D Complex conjugation of the Q_i functions

Another set of useful relations concerns the expected symmetry of the QSC system under complex conjugation. Let's assume that under the complex conjugation the equation (2.9) remain invariant. In general this is true if

$$\bar{\mathbf{P}}_a = \lambda_a^b \mathbf{P}_b, \quad \bar{\mathbf{P}}^a = \lambda^a_b \mathbf{P}^b \quad (\text{D.1})$$

for some constant coefficients λ_a^b , such that $\lambda_a^b \lambda^a_c = -\delta_c^b$ (in our case $\lambda_a^b = -i\delta_a^b$). If this holds the complex conjugate \bar{Q}_i should give an alternative complete set of solutions of the finite difference equation (2.9), which should be related to the initial set as a linear combinations with some i -periodic coefficients

$$\bar{Q}_i = \Omega_i^j Q_j, \quad \Omega_i^j(u+i) = \Omega_i^j(u). \quad (\text{D.2})$$

Those coefficients can be written in terms of $Q_{a|i}$ as

$$\Omega_i^j = -\bar{Q}_{a|i} \left(u - \frac{i}{2}\right) \lambda^a_b Q^{bj} \left(u - \frac{i}{2}\right) \quad (\text{D.3})$$

where $Q^{bj} = -((Q_{b|j})^{-1})^T$. We can easily check this is indeed true. We show that (D.2) holds:

$$\Omega_i^j Q_j = -\bar{Q}_{a|i} \left(u - \frac{i}{2}\right) \lambda^a_b Q^{bj} \left(u - \frac{i}{2}\right) Q_j = -\bar{Q}_{a|i} \left(u - \frac{i}{2}\right) \lambda^a_b \mathbf{P}^b = -\overline{Q_{a|i} \left(u + \frac{i}{2}\right) \mathbf{P}^a} = \bar{Q}_i \quad (\text{D.4})$$

and also that the r.h.s. (D.3) is periodic:

$$\bar{Q}_{a|i}^+ \lambda^a_b Q^{bj+} = \overline{(Q_{a|i}^+ - \mathbf{P}_a \mathbf{Q}_i)} \lambda^a_b (Q^{bj-} + \mathbf{P}^b \mathbf{Q}^j) = \bar{Q}_{a|i}^- \lambda^a_b Q^{bj-} \quad (\text{D.5})$$

(we denoted $f^\pm = f(u \pm i/2)$). Finally we can find discontinuity of Ω using this identity

$$\tilde{\Omega}_i^j - \Omega_i^j = -\bar{Q}_{a|i}^- \lambda^a_b (\tilde{Q}^{bj-} - Q^{bj-}) = \bar{Q}_{a|i}^- \lambda^a_b (\tilde{\mathbf{P}}^b \tilde{\mathbf{Q}}^j - \mathbf{P}^b \mathbf{Q}^j) = -\bar{\mathbf{Q}}_i \tilde{\mathbf{Q}}^j + \bar{\mathbf{Q}}_i \mathbf{Q}^j. \quad (\text{D.6})$$

We notice one more relation which we will use below. Consider Ω_1^4 . Its discontinuity is due to (2.12)

$$\tilde{\Omega}_1^4(u) - \Omega_1^4(u) = u \bar{\mathbf{q}}_1(u) \mathbf{q}_1(u) - u \bar{\mathbf{q}}_1(-u) \mathbf{q}_1(-u) \quad (\text{D.7})$$

from where we see that Ω_1^4 should be an even function.

E Expansion of $q_1(u)$ at the origin

As discussed in the end of section 4.2, to demonstrate that the Schrödinger equation is encoded in the QSC we need to compute the expansion of $q_1(u)$ at the origin up to the term linear in u . Let us show how this can be done.

On the one hand, from the 2nd order difference equation (4.5) on q_1 we find that $q_1(0)$ and $q_1'(0)$ are related to its expansion at $u = -i$:

$$q_1(u) = \frac{4\hat{g}^2}{(u+i)^2} q_1(0) + \frac{4\hat{g}^2 q_1'(0) - 2q_1(0)\Omega}{u+i} + \mathcal{O}((u-i)^0). \quad (\text{E.1})$$

On the other hand, we can compute the expansion around $u = -i$ using the expression (4.20) for q_1 in terms of $F(z)$. In that expression the singularity of q_1 at $u = -i$ arises because the integrand is singular when $z = i$. In the vicinity of this point $F(z)$ is a linear combination of two solutions of the the Schrödinger equation, one of which is smooth at $z = i$ and the other one also includes terms of the type $(z - i)^n \log(iz + 1)$ with $n \geq 1$. Solving the equation close to this point we find

$$F(z) = -\frac{iC}{2\hat{g}^2} + C(z - i) \log(iz + 1) + iC_2(z - i) + \dots, \quad (\text{E.2})$$

where the real⁷ constant C_2 comes from the smooth solution and dots stand for more regular terms. Let us also note that the expression (4.20) is not applicable directly for $\text{Im } u < 0$ as the integrand is too singular near $z = i$. However, as we need only the coefficients of the double and the single pole at $u = -i$ in $q_1(u)$, we can modify (4.20) in a way which ensures convergence of the integral without changing these two coefficients:

$$2 \int_i^\infty dz \frac{e^{-i\Omega/2} \left(\frac{z+i}{z-i} \right)^{iu}}{z^2 + 1} \left[\frac{iC}{2\hat{g}^2} + e^{-i\frac{z-\Omega}{2}} \left(-\frac{iC}{2\hat{g}^2} + C(z - i) \log(iz + 1) + iC_2(z - i) + \dots \right) \right]. \quad (\text{E.3})$$

We subtracted a part proportional to the integral

$$2 \int_i^\infty dz \frac{1}{z^2 + 1} \left(\frac{z+i}{z-i} \right)^{iu} = \frac{1}{u}, \quad (\text{E.4})$$

which does not affect the two coefficients we are after. From (E.3) we now find

$$q_1(u) = -\frac{2iCe^{-\frac{i\Omega}{2}}}{(u+i)^2} - \frac{e^{-\frac{i\Omega}{2}} \left(-\frac{iC\Omega}{2\hat{g}^2} - 2i\pi C - 2C - 2C \log 2 - 2iC_2 \right)}{u+i} + \mathcal{O}((u-i)^0). \quad (\text{E.5})$$

Comparing this with (E.1) we get

$$q_1(0) = -\frac{iCe^{-\frac{i\Omega}{2}}}{2\hat{g}^2}, \quad (\text{E.6})$$

$$q_1'(0) = \frac{e^{-\frac{i\Omega}{2}} (4\hat{g}^2 (iC_2 + C(1 + i\pi + \log 2)) - iC\Omega)}{8\hat{g}^4}. \quad (\text{E.7})$$

This finally allows to construct the combination $e^{2\pi u} q_1(u) \bar{q}_1(u)$ which we need. We observe that C_2 cancels out and we find

$$e^{2\pi u} q_1(u) \bar{q}_1(u) = -\frac{C^2}{4\hat{g}^4} - \frac{C^2 \Omega}{8\hat{g}^6} u + \mathcal{O}(u^2), \quad (\text{E.8})$$

which is the key result used in (4.31) in the main text.

⁷One can show that C_2 is real using the fact that $F(z)$ is a real and even function.

F Numerical data

Here we present a part of our numerical data for the quark-antiquark potential Ω at finite coupling g with zero twist $\theta = 0$. While the accuracy might vary slightly, we expect all digits to be correct (with uncertainty in the last digit).

| g | $\Omega(g)$ | g | $\Omega(g)$ |
|-------|-----------------------|------|-----------------------|
| 0 | 0 | 0.05 | 0.02937069654776 |
| 0.075 | 0.06265474565224 | 0.1 | 0.10511713720337 |
| 0.125 | 0.15465836443567 | 0.15 | 0.20955607216466 |
| 0.175 | 0.26845318866584 | 0.2 | 0.330312294925133 |
| 0.225 | 0.39435828555165 | 0.25 | 0.4600215248401101992 |
| 0.275 | 0.5268878004652301086 | 0.3 | 0.594657468302282222 |
| 0.325 | 0.6631138101939375140 | 0.35 | 0.7320994342456940408 |
| 0.375 | 0.8014991401020814198 | 0.4 | 0.8712277052055592640 |
| 0.425 | 0.9412212786455914103 | 0.45 | 1.0114313519742950991 |
| 0.475 | 1.0818205391585539063 | 0.5 | 1.1523596132795935855 |
| 0.525 | 1.2230254118796313025 | 0.55 | 1.2937993424631526624 |
| 0.575 | 1.3646663040854278314 | 0.6 | 1.4356138993330072537 |
| 0.625 | 1.5066318508313724359 | 0.65 | 1.5777115633938239593 |
| 0.675 | 1.6488457911511407735 | 0.7 | 1.7200283813289496328 |
| 0.725 | 1.7912540747152853534 | 0.75 | 1.8625183485921063555 |
| 0.775 | 1.9338172918626574100 | 0.8 | 2.0051475048695944173 |
| 0.825 | 2.0765060183502537912 | 0.85 | 2.1478902273706390145 |
| 0.875 | 2.2192978370894877695 | 0.9 | 2.2907268179434612004 |
| 0.925 | 2.3621753683925865485 | 0.95 | 2.4336418837756454947 |
| 0.975 | 2.5051249301358780128 | 1. | 2.5766232221146891288 |
| 1.025 | 2.6481356041939115734 | 1.05 | 2.7196610347092241265 |
| 1.075 | 2.7911985721684911266 | 1.1 | 2.8627473634963918184 |
| 1.125 | 2.9343066338961908685 | 1.15 | 3.0058756780749462488 |
| 1.175 | 3.0774538526229463147 | 1.2 | 3.1490405693740687669 |
| 1.225 | 3.2206352896028680533 | 1.25 | 3.2922375189379219065 |
| 1.275 | 3.3638468028903889215 | 1.3 | 3.4354627229126975867 |
| 1.325 | 3.5070848929154719081 | 1.35 | 3.5787129561817286982 |
| 1.375 | 3.6503465826264772085 | 1.4 | 3.7219854663574488362 |
| 1.425 | 3.793629323499052523 | 1.45 | 3.865277890247007235 |
| 1.475 | 3.936930921125622182 | 1.5 | 4.008588187423520918 |

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